Cottingham formula and the pion electromagnetic mass difference at finite temperature

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Abstract

We generalize the Cottingham formula at finite ($T \neq 0$) temperature by using the imaginary time formalism. The Cottingham formula gives the theoretical framework to compute the electromagnetic mass differences of the hadrons using a dispersion relation approach. It can be also used in other contexts, such as non leptonic weak decays, and its generalization to finite temperature might be useful in evaluating thermal effects in these processes. As an application we compute the $\pi^+ - \pi^0$ mass difference at $T \neq 0$; at small T we reproduce the behaviour found by other authors: $\delta m^2(T) = \delta m^2(0) + \mathcal{O}(\alpha T^2)$, while for moderate T, near the deconfinement temperature, we observe deviations from this behaviour.

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1 Introduction

The aim of this paper is to present the generalization of the Cottingham formula [1] to finite temperature (for a review of field theory at finite temperature and density see [2]). The Cottingham formula allows the evaluation of the time-ordered product of two hadronic currents between hadronic states by a Dispersion Relation (DR) and its main application is in the evaluation of electromagnetic mass differences of hadrons. The use of the Cottingham formula in this context predates Quantum Chromo Dynamics (QCD) [3]; by the advent of QCD some modifications of the original formalism were adopted: for example, the problem of the convergence was settled by the use of an ultraviolet cut-off μ^2 related to the renormalization procedure [4]. Furthermore, the use of effective field theories: chiral perturbation theory [5] and heavy quark effective theory [6], has introduced well defined theoretical schemes to evaluate the relevant Feynman diagrams.

Besides the use in the context of the electromagnetic mass differences of hadrons, the Cottingham formula has ben also applied in other fields, mainly related to non leptonic weak processes: kaon decays [7], [8], $K^0 - \bar{K}^0$ mixing [9], parity violations in nucleon interactions [10], B-meson processes [11]; therefore its extension to finite temperature may be of some interest to analyze the role of the thermal effects in these physical situations. Among the possible applications we shall pick up in this paper the $\pi^+ - \pi^0$ mass difference at finite temperature, a subject that has received a continuous attention both in the past [12] and in more recent times [13].

2 Cottingham formula at finite temperature

As already stressed in the introduction, the main application of the Cottingham formula is for the calculation of the electromagnetic mass splitting of mesons. The mass shift of the meson M due to the electromagnetic interaction can be obtained by computing:

$$\delta m^2 = \frac{ie^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{g_{\mu\nu}}{q^2+i\epsilon}T^{\mu\nu}(q,p)\ , \tag{1}$$

where the hadronic tensor

$$T^{\mu\nu}(q,p) = i\int d^4x\ e^{-iqx}\langle M(p)|T(J^\mu(x)J^\nu(0)|M(p)\rangle \tag{2}$$

describes the Compton scattering of a virtual photon of four momentum q^μ off the meson M of momentum p^μ and J^μ is the electromagnetic current. The

Compton amplitude can be decomposed in terms of gauge invariant tensors as follows:

$$T^{\mu\nu}(q,p) = D_1^{\mu\nu}\ T_1(q^2,\nu)\ +\ D_2^{\mu\nu}\ T_2(q^2,\nu)\ , \tag{3}$$

where

$$\begin{aligned} D_1^{\mu\nu} &= -g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\ , \\ D_2^{\mu\nu} &= \frac{1}{m^2}\left(p^\mu - \frac{\nu}{q^2}q^\mu\right)\left(p^\nu - \frac{\nu}{q^2}q^\nu\right), \\ \nu &= pq\ . \end{aligned} \tag{4}$$

The Lorentz invariant structure functions $T_1(q^2,\nu)$ and $T_2(q^2,\nu)$ depend, in the meson rest frame, by $|\vec{q}|$ and $q^0 = \frac{\nu}{m}$, where m is the meson mass. For fixed $|\vec{q}|$, the singularities in the complex q^0 plane are placed just below the positive real axis and just above the negative real axis; therefore one can perform a Wick rotation to the imaginary axis $q^0 = ik^0$, without encountering any singularity. After this transformation, the integration involves only space-like momenta for the photon:

$$\begin{aligned} q^0 &\rightarrow ik^0, \\ q^2 &\rightarrow -Q^2 = -(k_0^2 + |\vec{q}|^2)\ . \end{aligned} \tag{5}$$

After the change of variables (5) one obtains the Cottingham formula [1]:

$$\begin{aligned} \delta m^2 &= \frac{e^2}{16\pi^3}\int_0^{+\infty}\frac{dQ^2}{Q^2}\int_{-\sqrt{Q^2}}^{+\sqrt{Q^2}} dk^0\sqrt{Q^2-k_0^2}\ \times \\ &\times \left[-3T_1(-Q^2, imk^0) + \left(1-\frac{k_0^2}{Q^2}\right)T_2(-Q^2, imk^0)\right]\ . \end{aligned} \tag{6}$$

As discussed in [4] and [6], the Q^2 integration range is cut-off at $Q^2_{max} = \mu^2$. μ is the Quantum Chromo Dynamics (QCD) renormalization mass scale, at which both the strong coupling constant α_s and the quark masses m_q have to be specified and roughly represents the onset of the scaling behaviour of QCD. As is well known the renormalization procedure introduces counterterms which cancel the infinite contribution induced by virtual particles with momenta larger than μ: therefore its net effect is analogous to a cut-off of the Q^2 integral at μ^2. For approximate calculations there is a residue smooth dependence on μ, which in a complete calculation is exactly canceled by the $\mu-$dependence of the renormalized quark masses and strong coupling constant. Typical values of μ are in the range of 1-3 GeV, corresponding to the onset of the scaling behaviour of QCD and to a mass scale significantly larger than all the hadronic masses.

We wish now to generalize this formula at finite temperature; we choose to work in the imaginary time formalism, which corresponds to substitute the integral over the energies with a discrete sum over the so called *Matsubara frequencies* $\omega_n = 2\pi n T$:

$$\int_{-\infty}^{+\infty} dq^0 f(q^0) \rightarrow 2\pi T \sum_{n=-\infty}^{+\infty} f(q^0)\Big|_{q^0= i\ \omega_n} . \tag{7}$$

This substitution corresponds to insert in (6) the factor

$$2\pi T \sum_{n=-\infty}^{+\infty} \delta(k^0 - \omega_n) . \tag{8}$$

We therefore obtain, from (6) and (8):

$$\begin{aligned} \delta m^2 &= \frac{\alpha}{4\pi^2} \int_0^{+\mu^2} \frac{dQ^2}{Q^2} 2\pi T \sum_{n=-[a]}^{+[a]} \sqrt{Q^2 - \omega_n^2} \times \\ &\times \left[-3T_1(-Q^2,\, i\, m\, \omega_n) + \left(1 - \frac{\omega_n^2}{Q^2}\right) T_2(-Q^2,\, i\, m\, \omega_n) \right] , \end{aligned} \tag{9}$$

where

$$a = \frac{\sqrt{Q^2}}{2\pi T} . \tag{10}$$

$[a]$ represents the maximum integer contained in the real number a.

Eq.(9) generalizes the Cottingham formula at $T \neq 0$ and is the starting point of the application to the $\pi^+ - \pi^0$ mass difference at finite temperature to be discussed in the following section.

3 The pion electromagnetic mass difference at finite temperature

The invariant amplitudes T_1 and T_2 satisfy dispersion relations (DR) in the ν variable. The DR for T_2 is unsubtracted, while T_1 requires one subtraction[3], [5]:

$$T_1(q^2, \nu) = T_1(q^2, 0) + \frac{\nu^2}{\pi} \int_0^{+\infty} \frac{d\nu'^2}{\nu'^2} \frac{Im\ T_1(q^2, \nu')}{\nu'^2 - \nu^2}, \tag{11}$$

$$T_2(q^2, \nu) = \frac{1}{\pi} \int_0^{+\infty} d\nu'^2 \frac{Im\ T_2(q^2, \nu')}{\nu'^2 - \nu^2} . \tag{12}$$

After the change of variables (5) these equations become:

$$T_1(-Q^2, i\,m\,k^0) \;=\; T_1(-Q^2, 0) \;-\; m^2 k_0^2 \int_0^{+\infty} \frac{d\nu'^2}{\nu'^2}\,\frac{W_1(-Q^2,\nu')}{\nu'^2 + m^2\,k_0^2}, \tag{13}$$

$$T_2(-Q^2, i\,m\,k^0) \;=\; \int_0^{+\infty} d\nu'^2\,\frac{W_2(-Q^2,\nu')}{\nu'^2 + m^2\,k_0^2}, \tag{14}$$

$$W_i(q^2,\nu) \;=\; \frac{1}{\pi}\, Im\; T_i(q^2,\nu)\;,\;\;(i=1,2). \tag{15}$$

In order to evaluate the DR in the case of the pion electromagnetic mass difference we consider the contribution of the Born term (the π meson itself) and the $J^P = 1^-$ resonances ω. Other contributions might be in principle sizeable, however it is well known, since the work by Harari [3], that the DR are sufficiently well convergent at $T=0$ and the first two polar terms in (11) represent by far the largest contribution to the electromagnetic mass difference. This stems from the fact that the $\pi^+ - \pi^0$ mass difference takes contribution from the Isospin= 2 mass term, and the time ordered product of the two electromagnetic currents in this case has no singularities for $x \to 0$ (or, equivalently, for $Q^2 \to \infty$), differently from other hadronic mass differences, such as the proton-neutron mass difference, that present a $1/x^2$ light-cone singularity. For the same reason the limit $\mu \to \infty$ could also be taken in this case.

To compute the different contributions to (11) we consider the following matrix elements ($q = p' - p$):

$$< \pi^+(p')\;|J^\mu_{\rm em}\;|\;\pi^+(p) > \;=\; F(q^2)\;(p+p')^\mu \tag{16}$$

$$< \omega(p',\epsilon)\;|J^\mu_{\rm em}\;|\;\pi^0(p) > \;=\; i\;h(q^2)\;\epsilon^{\mu\lambda\rho\sigma}\epsilon^*_\lambda\;q_\rho\;p_\sigma \tag{17}$$

where ϵ_λ is the ω polarization vector and F, h are electromagnetic form factors. They can be written as follows:

$$F(q^2) \;=\; \frac{1}{1 - q^2/m_V^2} \tag{18}$$

$$h(q^2) \;=\; \frac{h(0)}{1 - q^2/m_V^2} \tag{19}$$

m_V is a mass parameter that we identify with the ρ mass. Indeed both form factors can be obtained by assuming ρ dominance. This hypothesis allows to extrapolate the behaviour at finite temperature. Indeed at $T \neq 0$, but small, one has [14]

$$F(q^2,T) \;=\; \left[1 + \frac{2T^2}{3f_\pi^2} g_0\left(\frac{m_\pi^2}{T^2}\right)\right] \times \times \left[\frac{g_{\rho\pi\pi}(T) g_{\rho\gamma}(T)}{m_\rho^2}\frac{1}{1-q^2/m_\rho^2} \;-\; \frac{T^2}{4f_\pi^2} g_0\left(\frac{m_\pi^2}{T^2}\right)\right]\;. \tag{20}$$

This expression takes into account the effective charge of a pion in thermal medium containing other pions at the equilibrium. Here $f_\pi = 93$ MeV, $g_0(x)$ is given by [14]:

$$g_0(x) = \frac{1}{2\pi^2} \int_0^\infty dy \frac{y^2}{\sqrt{x^2+y^2}\exp(\sqrt{x^2+y^2})-1)} \ , \tag{21}$$

while $g_{\rho\pi\pi}(T)$ and $g_{\rho\gamma}(T)$ are coupling constants for the $\rho\pi\pi$ and $\rho-$photon vertex respectively, computed taking into account the effects of the thermal bath. They are obtained in the soft pion limit, which is sufficiently accurate for our purposes [14]. These effects (and the related $f_\pi(T)$ dependence) have been computed by several authors and with different methods (see, for example [14],[15] and references therein). The results are:

$$g_{\rho\pi\pi}(T) \ = \ g_{\rho\pi\pi}\left[1 - \frac{5T^2}{12f_\pi^2} g_0\left(\frac{m_\pi^2}{T^2}\right)\right] \tag{22}$$

$$g_{\rho\gamma}(T) \ = \ g_{\rho\gamma}\left(1 - \frac{T^2}{12f_\pi^2}\right) \tag{23}$$

The values of these constants at $T = 0$ are related by the formula

$$\frac{g_{\rho\pi\pi}g_{\rho\gamma}}{m_\rho^2} = 1 \ , \tag{24}$$

a relation that implements the idea of ρ dominance of the electromagnetic form factor. It is worth stressing that, independently of the thermal effects induced by electromagnetism, the pion mass gets a thermal self energy also by the strong interactions with the dilute pion gas. In general we shall not consider these effects since they are identical for π^+ and π^0 and cancel in the difference. In one case the thermal self energy acts however as an infrared regulator (see below). We quote therefore this effect [13]:

$$m_{\pi^{+,0}}^2(T) = m_{\pi^{+,0}}^2\left(1 - \frac{T^2}{6f_\pi^2}\right) \tag{25}$$

We also observe that, because of the interaction with the thermal bath, pions can couple directly to the photon without the intermediate virtual ρ state: this effect is at the origin of the last term in (20). This term, which only exists at $T \neq 0$, gives a non-vanishing contribution to the form factor for $Q^2 \to +\infty$, which might be an artifact of the effective theory employed to compute it. However, as the Q^2 integration is cut-off at μ^2 and the results depends smoothly on μ, there is no need to correct for this effect and we shall assume (20) as it stands.

Let us now turn to the form factor relative to the ω intermediate state. We put $h(0) = 2.6\ \mathrm{GeV}^{-1}$, which can be derived by $\omega \to \pi\gamma$ decay rate. As it will be clear in the sequel, a more precise determination of $h(q^2)$ is not necessary because of the smaller role played by the ω intermediate state in the calculation. For the same reason we omit to include thermal effects in the vertex $\omega\pi\gamma$ that, to our knowledge, have not yet been computed.

Using the matrix elements and the coupling constants already introduced, we can calculate the electromagnetic pion mass splitting. To start with, we consider the $T = 0$ case. The contributions of the different terms to the DR are as follows; the subtraction term $T_1(q^2, 0)$ is given by:

$$T_1(q^2, 0) = -2F^2(q^2) + \frac{m^2 q^2 h^2(q^2)}{\nu_R}\ . \tag{26}$$

The two structure functions $W_{1,2}(q^2, \nu)$ that appear in (13) and (14) are given by:

$$W_1(q^2, \nu) = +\nu_R \left(m^2 q^2 - \nu_R^2\right) h^2(q^2)\ \delta\left(\nu^2 - \nu_R^2\right) \tag{27}$$

$$\begin{aligned} W_2(q^2, \nu) &= -2m^2 q^2 F^2(q^2)\ \delta\left(\nu^2 - \frac{q^4}{4}\right) \\ &+ q^2 m^2 \nu_R\ h^2(q^2)\ \delta\left(\nu^2 - \nu_R^2\right) \end{aligned} \tag{28}$$

where $\nu_R = \dfrac{m_\rho^2 - m^2 - q^2}{2}$. It is worth stressing that the Born term contributes only to $W_2(q^2, \nu)$, its contribution to T_1 only being through $T_1(q^2, 0)$.

These expressions can be used to compute the electromagnetic pion mass difference at $T = 0$. The results are reported in Table I, for $\mu = 2$ and $\mu = 3$ GeV and for $\mu = \infty$. It may be useful to stress that the numerical results are remarkably insensitive to variations of the cut-off. They show show that the pion contribution represents by far the dominant part of the mass difference. The small difference (around 10%) between the data and the theoretical result should be attributed to a few other poles in the dispersion relation, most notably the a_1 resonance.

Contribution	$\mu =$ 2 GeV	$\mu =$ 3 GeV	$\mu = \infty$
π	3.56	3.82	4.05
ω	0.09	0.11	0.13
Total	3.65	3.93	4.18

Table I. Contributions to δm in MeV; $\delta m_{exp} = 4.6$ MeV.

In the chiral limit $m_\pi = 0$ we get, for $\mu = \infty$:

$$\delta m^2 = \frac{3\,\alpha\, m_\rho^2}{4\,\pi}\ , \tag{29}$$

which gives, numerically, $\delta m = m_{\pi^+} - m_{\pi^0} = 3.8$ MeV, to be compared to the the experimental result $\delta m_{exp} = 4.6$ MeV. We can also compare it to the the result of Das et al. [16]:

$$\delta m^2 = \frac{3\,\alpha\, m_\rho^2}{4\,\pi}\frac{f_\rho^2}{f_\pi^2}\ln\Big(\frac{f_\rho^2}{f_\rho^2 - f_\pi^2}\Big) \tag{30}$$

that is obtained using the chiral limit and the Weinberg sum rules. The extra factor in (30) as compared to (29) is given by $\dfrac{f_\rho^2}{f_\pi^2}\ln\Big(\dfrac{f_\rho^2}{f_\rho^2 - f_\pi^2}\Big)$ and is numerically equal to 1.24 for $\dfrac{f_\rho}{f_\pi} = 1.66$.

Let us now consider the analogous calculation at $T \neq 0$. We shall use in (9)

$$\begin{aligned} T_1(-Q^2, i\,m\,\omega_n) &= -2F^2(-Q^2,T) \\ &- m^2 h^2(-Q^2)\Big[\frac{2Q^2}{Q^2+m_\omega^2-m^2} - \frac{\omega_n^2(\nu_R^2+m^2Q^2)}{\nu_R(\nu_R^2+m^2\omega_n^2)}\Big] \end{aligned} \tag{31}$$

$$\begin{aligned} T_2(-Q^2, i\,m\,\omega_n) &= 8m^2F^2(-Q^2,T)\frac{Q^2-m^2+m^2(T)}{\Big((Q^2-m^2+m^2(T)\Big)^2+4m^2\omega_n^2} + \\ &- m^2h^2(-Q^2)\frac{Q^2\nu_R}{\nu_R^2+m^2\omega_n^2}\ , \end{aligned} \tag{32}$$

where $m^2(T)$ is given in (25). By inserting these results in eq.(9) we can compute numerically the sums over the discrete energy. The numerical results will be discussed in the next Section.

4 Electromagnetic mass difference in the chiral limit

In the chiral limit the expression for the electromagnetic pion mass difference looks remarkably simple:

$$\delta m^2(T) = \frac{6\alpha}{4\pi^2}\int_0^{\mu^2}\frac{dQ^2}{Q^2}F^2(-Q^2,T)\ 2\pi\ T\sum_{n=-[a]}^{+[a]}\sqrt{Q^2-\omega_n^2}\ . \tag{33}$$

The behaviour of $\delta m^2(T)$ at small T is as follows:

$$\delta m^2(T) = \delta m^2 + \lambda T^2 \ , \tag{34}$$

where $\delta m^2 = \delta m^2(T=0)$ and λ is a coefficient.

The absence of the term linear in T can be proved explicitly by performing an asymptotic expansion in $1/T$ of $\delta m^2(T)$. The term independent of T is given by

$$\begin{aligned}\lim_{T\to 0}\delta m^2(T) &= \frac{6\alpha}{4\pi^2}\int_0^{\mu^2} dQ^2 F^2(-Q^2) \lim_{a\to\infty}\frac{1}{a}\sum_{n=-[a]}^{+[a]}\sqrt{1-\frac{n^2}{a^2}} = \\ &= \frac{6\alpha}{4\pi^2}\int_0^{\mu^2} dQ^2 F^2(-Q^2)\int_{-1}^{+1} dx\sqrt{1-x^2} = \\ &= \frac{3\alpha}{4\pi}\int_0^{\mu^2} dQ^2 F^2(-Q^2) \ ,\end{aligned} \tag{35}$$

which coincides with δm^2, i.e. with the result obtained by putting directly $T=0$ from the very beginning. As to the coefficient of the term in T^2, let us write it as follows:

$$\lambda = \mathcal{L}_1 \ + \ \mathcal{L}_2 \ , \tag{36}$$

as it arises from two different sources. The first term, $\mathcal{L}_1$, is obtained by expanding $F^2(-Q^2,T)$ in T and taking into account that $g_0(0)=1/12$:

$$F^2(-Q^2,T) = F^2(-Q^2) - \frac{T^2}{8f_\pi^2}\Big[F^2(-Q^2) + \frac{F(-Q^2)}{3}\Big] + \mathcal{O}(T^2) \ ; \tag{37}$$

from this equation one gets the following contribution to $\delta m^2(T)$

$$\delta m^2(0)\ \Big[1 - \frac{T^2}{8f_\pi^2}\Big(1+\frac{m_\rho^2+\mu^2}{3\mu^2}\ln\frac{m_\rho^2+\mu^2}{m_\rho^2}\Big)\Big] \ , \tag{38}$$

and, therefore,

$$\mathcal{L}_1 = -\frac{T^2}{8f_\pi^2}\delta m^2(0)\ \Big(1+\frac{m_\rho^2+\mu^2}{3\mu^2}\ln\frac{m_\rho^2+\mu^2}{m_\rho^2}\Big) \ . \tag{39}$$

The second term, i.e. $\mathcal{L}_2$, is obtained by putting

$$F^2(-Q^2,T) = F^2(-Q^2) \tag{40}$$

and computing

$$\mathcal{L}_2 = \lim_{T\to 0}\ \frac{6\alpha}{4\pi^2 T^2}\int_0^{\mu^2} dQ^2 F^2(-Q^2)\Big[\frac{1}{a}\sum_{n=-[a]}^{+[a]}\sqrt{1-\frac{n^2}{a^2}} - \frac{\pi}{2}\Big] \ , \tag{41}$$

where, as before, $a = \frac{\sqrt{Q^2}}{2\,\pi\,T}$. We have been unable to evaluate analytically this limit; however, when calculated numerically, for $\mu \geq 0.5 - 1$ GeV, the limit is basically independent of μ and within a few percent is given by:

$$\mathcal{L}_2 \approx \pi\alpha \ . \tag{42}$$

Putting the two contributions together we get

$$\delta m^2(T) \approx \delta m^2(0)\ \Big[1 - \frac{T^2}{6 f_\pi^2}\Big(\frac{3}{4} + \frac{m_\rho^2+\mu^2}{4\mu^2}\ln\frac{m_\rho^2+\mu^2}{m_\rho^2}\Big)\Big]\ +\ \pi\alpha T^2\ . \tag{43}$$

We observe however that the behaviour of the last term in eq. (43) holds only at small T ($T \leq 100$ MeV). For $T \geq \frac{\mu}{2\pi}$, its T-dependence would be linear:

$$\frac{3\alpha T}{\pi}\int_0^{\mu^2}\frac{dQ^2}{\sqrt{Q^2}}F^2(-Q^2)\ , \tag{44}$$

but, given the range of the possible values of μ, such a behaviour cannot be reached, as the deconfinement process should take place at much smaller value of T. For $T > 100$ MeV numerical deviations from the behaviour (43) are therefore expected, as we will discuss below.

We can compare these results with those obtained by the authors of [13] in the framework of chiral perturbation theory at finite temperature within the hard thermal loop approximation:

$$\delta m^2(T) = \delta m^2(0)\ \Big[1 - \frac{T^2}{6 f_\pi^2}\Big]\ +\ \pi\alpha T^2\ . \tag{45}$$

A part from the difference in $\delta m^2(0)$ discussed above, we get a small deviation also in the correction $\propto T^2$: whereas the term $+\pi\alpha T^2$ is identical, the remaining part $\propto T^2$ differs by the factor

$$\frac{3}{4} + \frac{m_\rho^2+\mu^2}{4\mu^2}\ln\frac{m_\rho^2+\mu^2}{m_\rho^2}\ , \tag{46}$$

whose numerical value is 1.3 at $\mu = 2$ GeV and 1.5 at $\mu = 3$ GeV. This difference arises from the different treatment of the virtual photon effect: in particular the logarithmic μ dependence in (43) arises from the direct photon coupling to pions, which is allowed by the pion electromagnetic form factor at $T \neq 0$, as given by (20). In particular if one assumes that also this direct coupling, i.e. the last term proportional to T^2 in (20), is multiplied by the factor $\left(1 - \frac{q^2}{m_\rho^2}\right)^{-1}$, then the same result as in [13] is obtained.

Let us now turn to the numerical results of our analysis. In Fig. 1 we report two curves. The solid line is the result of the full numerical analysis contained in Section 3, while the dashed line represents $\delta m^2(T)$ as computed by the approximate formula (43). Both curves are obtained at $\mu = 3$ GeV. The comparison shows that:
1) For small T, the small deviation ($< 7\%$) is mainly due to a chiral symmetry breaking term which is taken into account in the full calculation (solid line) and not in the approximate calculation (dashed line). This extra term is part of the pion pole contribution to the dispersion relation (the ω contribution is always very tiny).
2) For large T, i.e. $T > 100$ MeV, the difference is mainly due to the fact that the approximation (43) gets worse with the increasing temperature; in particular the positive contribution, i.e. the last term in (43), is quadratic only for $T \leq 100$ MeV, while for $T > 100$ MeV, it increases more slowly and the effect of the negative term remains unbalanced. For $T = 200$ MeV this correction is of the order of 80%.

All these results are almost independent of μ in the range $2 - 3$ GeV.

5 Conclusions

We have extended the Cottingham formula at finite temperature. This formula allows the computation of time ordered products of two currents between hadronic states; even though it is generally applied to the calculation of electromagnetic mass differences of hadrons, its possible range of applications is wider and includes the evaluation of matrix elements of other products of currents such as those occurring in the analysis of weak non leptonic decays. Therefore its extension to $T \neq 0$ may permit the study of finite temperature effects also for weak hadronic processes. As an application, we have considered the finite temperature $\pi^+ - \pi^0$ electromagnetic mass difference; the use of the Cottingham formula leads to results similar to those reached by chiral perturbation theory for small temperatures < 100 MeV. We have also computed deviations due to chiral symmetry breaking, which are of the order of 7% or less, and corrections to the hard thermal loop approximation whose role is relevant for $T \geq 100$ MeV.

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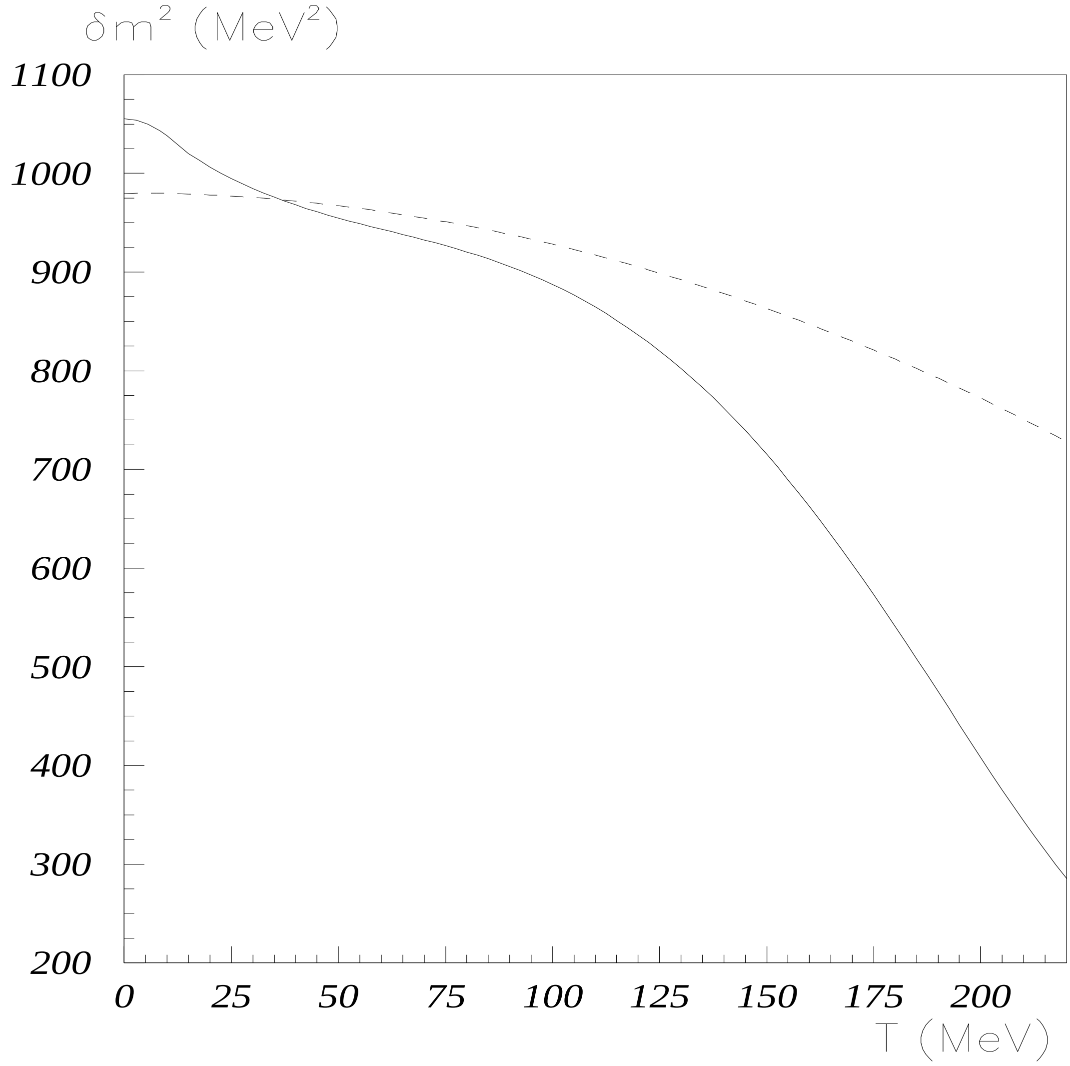

Figure caption

Fig. 1. The difference $\delta m^2 = m^2_{\pi^+} - m^2_{\pi^0}$ as a function of the temperature. The solid line is the result of the full calculation (with a value of the ultraviolet cut-off $\mu = 3$ GeV), the dashed line gives the result in the chiral and small temperature limit.